Notes on Financial Laws

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September 12th, 2017
0.1. Accumulation and discount processes

Financial calculus deals with money exchanges rolling over time at deferred dates. Two processes are considered:

- accumulation process
- discount process

An accumulation process transfers money forward in time (see Figure 1):

![Figure 1](image)

where:

- $C$ = principal or invested capital
- $M$ = final value (amount collectible at maturity)
- $\frac{M}{C} = f = \text{accumulation factor}$
- $M - C = I = \text{interest}$

Vice versa, a discount process transfers money backward in time (see Figure 2):

![Figure 2](image)
where:

\[
S = \text{nominal value (future amount)} \\
A = \text{present value (immediate amount)} \\
\frac{A}{S} = \phi = \text{discount factor} \\
S - A = D = \text{discount}
\]

In an \textit{accumulation process} the basic relation between the principal and the final value is:

\[
M = C \cdot f
\]

So, the interest:

\[
I = M - C = C \cdot f - C = C \cdot (f - 1)
\]

In a \textit{discount process}, the fundamental relation between the present value and the nominal value is:

\[
A = S \cdot \phi
\]

So, the discount:

\[
D = S - A = S - S \cdot \phi = S \cdot (1 - \phi)
\]

0.2. \textbf{Financial laws}

The exchange factors \(f\) and \(\phi\) are called \textit{financial factors}. In general, they are two-variable functions \(f(t, \alpha)\) and \(\phi(t, \beta)\) depending on the lifetime \(t\) and the \textit{cost of capital} \(\alpha\) (or \(\beta\)). As far as \(\alpha\) (or \(\beta\)) is fixed, \(f(t)\) and \(\phi(t)\) are called \textit{financial laws} (accumulation and discount financial law, respectively).

By means of \(f\) and \(\phi\) we can transfer money ”back and forth” over time. If relation

\[
f \cdot \phi = 1
\]

holds, \(f\) and \(\phi\) are said to be \textit{conjugate}. Then

\[
f = \frac{1}{\phi} \quad \quad \quad \phi = \frac{1}{f}
\]

Now, we define the \textit{interest rate} \(i\) as the interest earned on an unitary amount (for example, 1\(\text{€}\)) invested for one period time (for example, from year \(t = 0\) to year \(t = 1\)). So, we get

\[
I = M - C = 1 \cdot f(1) - 1 = i
\]

Analogously, the \textit{discount rate} \(d\) is the discount (= the reward) for discounting an unitary amount (for instance 1\(\text{€}\)) collecting at maturity 1 (for instance, receivable one year later):

\[
D = S - A = 1 - 1 \cdot \phi(1) = d
\]

So, we get:

\[
f(1) = 1 + i \quad \quad \quad \phi(1) = 1 - d
\]
Moreover, if the financial factors are conjugate:

\[ f(1) \cdot \phi(1) = 1 \]

holds, above can be rewritten as:

\[ (1 + i) \cdot (1 - d) = 1 \]

so

\[ d = \frac{i}{1 + i} \quad i = \frac{d}{1 - d} \]

Above shows the linkage between the interest rate \( i \) and the discount rate \( d \).

In the following we focus on the most common financial laws:

- simple accumulation (and simple discount)
- compound accumulation (and compound discount)
- anticipated simple interest accumulation (and bank discount)

### 0.2.1. Simple accumulation

The simple accumulation financial law can be characterized by the following statement: "the interest is proportional to the principal \( C \) and the lifetime \( t \) of the investment", i.e.

\[ \frac{I}{Ct} = \alpha \Rightarrow I = Ct\alpha \]

where \( \alpha \) is a (positive) constant of proportionality. Setting \( C = 1 \) and \( t = 1 \) we get:

\[ I = \alpha \]

so, \( \alpha \) is nothing but the accrued interest on a money unit over a time unit (usually the time unit is the year and the money unit is 1\( \text{€} \)), i.e., the (annual) simple interest rate \( i \). In conclusion, we get the basic formula:

\[ I = Cti \]

Therefore:

\[ M = C + I = C + Cti = C(1 + it) \]

and

\[ f(t, i) = 1 + it \quad \phi(t, i) = \frac{1}{1 + it} \]

where \( f(t, i) \) is the accumulation factor in the simple interest accumulation process, while \( \phi(t, i) \) is the conjugate discount factor in the simple discount (or rational discount) process.
Time unit changing

If time is measured in months, lifetime $t$ (in years) has to be multiplied by 12, so we get $12t$ months. Vice versa, if time is measured in years, lifetime $t$ (given in months) has be divided by 12. In conclusion, let $i$ be the annual interest rate, then the equivalent period interest rate $i_m$ (with reference to $\frac{1}{m}$ of a year) produces the same final value:

$$1 + it = 1 + i_mmt$$

So we get

$$i_m = \frac{i}{m} \quad i = m \cdot i_m$$

The rates $i$ and $i_m$ are called equivalent rates (in the simple interest accumulation).

**Example 1** Bob invests 10000€ for 15 months in a fund. If the fund remunerates at annual simple interest rate 10%, compute the final amount available.

Note that

$$M = C(1 + it)$$

since the rate is annual, the timelife must be expressed in portions of a year, so

$$M = 10000 \left(1 + 0.10 \cdot \frac{15}{12}\right) = 10000 \cdot 1.125 = 11250$$

**Example 2** Compute the present value of 15000€ with maturity 8 months, at annual simple interest rate 6%.

Let

$$A = \frac{S}{1 + it}$$

Since the rate is annual, lifetime $t$ must be expressed in a portion of a year:

$$A = \frac{15000}{1 + 0.06 \cdot \frac{8}{12}} = \frac{15000}{1.04} = 14423.08$$

**Example 3** Compute the quarterly simple interest rate equivalent to the annual simple interest rate 16%.

Let

$$i_m = \frac{i}{m}$$

since a quarter corresponds to $\frac{1}{4}$ of a year, the quarterly interest rate $i_4$ results:

$$i_4 = \frac{i}{4} \Rightarrow i_4 = \frac{0.16}{4} = 0.04$$

i.e. the quarterly simple interest rate equivalent to the 16% annual simple interest rate is 4%.
Example 4  Bob is wishing to invest 5000 € for 2 years. He faces two options: investing at 3% quarterly simple interest rate or at 3.5% 4-month period simple interest rate. Which one is more convenient for him?

Investing at 3% quarterly interest rate, after two years (= 8 quarters) you get:

\[ M = 5000 \left(1 + 0.03 \cdot 8\right) = 6200 \]

On the other hand, investing at 3.5% 4-month period interest rate, after two years (= 6 periods of 4-month length) is:

\[ M' = 5000(1 + 0.035 \cdot 6) = 6050 \]

and since \( M > M' \), the former is more convenient than the latter.

Another process of calculation can be followed. The annual interest rate equivalent to the 3% quarterly interest rate is:

\[ i = m \cdot i_m \Rightarrow i = 4 \cdot i_4 = 4 \cdot 0.03 = 0.12 \]

The annual interest rate equivalent to the 3.5% 4-month period interest rate is:

\[ i' = m' \cdot i_m \Rightarrow i' = 3 \cdot i_3 = 3 \cdot 0.035 = 0.105 \]

Since \( i > i' \), it is worthwhile investing at 3% quarterly interest rate.

0.2.2. Compound interest accumulation

In this accumulation process, interests earned over one period are immediately invested (the so-called “interest capitalization” is assumed). The accumulation process works like that:

\[
\begin{align*}
  t &= 0 & C \\
  t &= 1 & M = C(1 + i) \\
  t &= 2 & M = C(1 + i)(1 + i) = C(1 + i)^2 \\
  \cdots & \cdots & \cdots \\
  t &= n & M = C(1 + i)(1 + i)\ldots = C(1 + i)^n
\end{align*}
\]

In general, after \( t \) periods we have on hand:

\[ M = C(1 + i)^t \]

with \( t \in \mathbb{R} \) (\( t \) has not to be necessarily integer). Therefore:

\[ f(t, i) = (1 + i)^t, \quad \phi(t, i) = (1 + i)^{-t} \]

are the accumulation and discount conjugate factors in the compound interest accumulation. The rate \( i \) is called the compound interest rate.
## 0.2. Financial laws

### Time unit changing

Let $i$ be an annual interest rate. The period interest rate $i_m$ (with reference to $\frac{1}{m}$ of a year) stems from the equality in final value:

$$C(1 + i)^t = C(1 + i_m)^{mt}$$

where $C$ is the capital invested at epoch 0. So,

$$i_m = \sqrt[m]{1 + i} - 1$$

$$i = (1 + i_m)^m - 1$$

where $i$ and $i_m$ are equivalent rates (in the compound interest accumulation).

In real situations, another rate is commonly used. The $m$-convertible annual nominal interest rate $j_m$, that is obtained multiplying the period interest rate by the number of periods:

$$j_m = m \cdot i_m$$

The annual rate $i$, the annual nominal rate $j_m$ and the period rate $i_m$ are connected by:

$$j_m = m \left[ \sqrt[m]{1 + i} - 1 \right]$$

$$i = \left( 1 + \frac{j_m}{m} \right)^m - 1$$

### Example 5

Compute the final value obtained investing 1000€ for 5 years charging 8% annual compound interest.

We have:

$$M = C(1 + i)^t$$

and then:

$$M = 1000(1 + 0.08)^5 = 1000 \cdot 1.46933 = 1469.33$$

### Example 6

Compute the present value of 500€ with maturity 3 years and 4 months charging 9% annual compound interest.

Assuming the exponential convention, so that the discount factor $(1 + i)^{-t}$ is used also for periods of time $t$ that are not integer

$$A = S(1 + i)^{-t}$$

and then (expressing the time in years, since the rate is annual) we get:

$$A = 500(1 + 0.09)^{-\left(3 + \frac{4}{12}\right)} = 500 \cdot (1.09)^{-3.33} = 500 \cdot 0.75031 = 375.16$$

### Example 7

Compute the annual interest rate and the 12-convertible annual nominal interest rate equivalent to the 1% monthly interest rate.
Since  
\[ i = (1 + i_m)^m - 1 \quad \text{and} \quad j_m = m \cdot i_m \]
then:
\[ i = (1 + i_{12})^{12} - 1 \Rightarrow i = (1 + 0.01)^{12} - 1 = 0.1268 \]
\[ j_{12} = 12 \cdot i_{12} \Rightarrow j_{12} = 12 \cdot 0.01 = 0.12 \]
and therefore the annual interest rate equivalent to the 1% monthly interest rate is 12.68% and the 12-convertible annual nominal interest rate equivalent to the 1% monthly interest rate, is 12%.

0.2.3. Anticipated simple interest accumulation - Linear Discount

This accumulation process is defined as the conjugate of the bank discount accumulation. In such a case, "the discount is proportional to the nominal value \( S \) and to the lifetime \( t \)”, i.e.:
\[ \frac{D}{S_t} = \alpha \Rightarrow D = S \alpha \]
where \( \alpha \) is a (positive) constant of proportionality. Setting \( S = 1 \) and \( t = 1 \) we get:
\[ D = \alpha \]
where \( \alpha \) is the discount (=reward) deducted from one money unit receivable one period later, i.e. the (bank) discount rate \( d \). Therefore:
\[ D = S td \]
and also:
\[ A = S - D = S - S td = S(1 - dt) \quad \text{with} \quad dt < 1 \quad \text{i.e.} \quad t < \frac{1}{d} \]
from which:
\[ \phi(t, d) = 1 - dt \quad \text{and} \quad f(t, d) = \frac{1}{1 - dt} \]
where \( \phi(t, d) \) is the discount factor in the bank discount accumulation. Whereas, \( f(t, d) \) is the conjugate accumulation factor, the so-called
It is worthwhile warning to guaranteeing positiveness \( f(t, d) \), it must be \( dt < 1 \).

**Example 8** Compute the final value obtained investing 500€ for 5 months, in the anticipated simple interest accumulation, at 7% annual discount.

Since
\[ M = \frac{C}{1 - dt} \]
expressing the time in years, since the interest rate is annual, we get

\[
M = \frac{500}{1 - 0.07 \cdot \frac{5}{12}} = \frac{500}{0.97083} = 515.02
\]

**Example 9** Compute the present value of 1500€ with maturity 7 months, in the bank discount accumulation, at 5% annual discount.

We have:

\[
A = S(1 - dt)
\]

and then (expressing the time in years, since the rate is annual):

\[
A = 1500 \left(1 - 0.05 \cdot \frac{7}{12}\right) = 1500 \cdot 0.97083 = 1456.25
\]

**Example 10** Compute the final value obtained investing 100€ for 1 year and 6 months, in the anticipated simple interest accumulation, at 9% annual interest.

We have:

\[
M = \frac{C}{1 - dt}
\]

First, we have to carry out \(d\) by means of the relation:

\[
d = \frac{i}{1 + i} \Rightarrow d = \frac{0.09}{1.09} = 0.0825
\]

so

\[
M = \frac{100}{1 - 0.0825 \cdot 1.5} = \frac{100}{0.87614} = 114.14
\]

### 0.3. Exercises

**Determine the following equivalent rates:**

1) The quarterly interest rate equivalent to the 16% annual interest rate in the simple accumulation.

2) The quarterly interest rate and the 4-convertible annual nominal interest rate equivalent to the 12% annual interest rate in the compound accumulation.

3) The bimestrial discount rate equivalent to the 12% annual discount rate in the anticipated simple interest accumulation.

4) The bimestrial interest rate equivalent to the 1% monthly interest rate in the simple interest accumulation.

5) The monthly interest rate equivalent to the 2% bimestrial interest rate in the compound interest accumulation.

6) The annual discount rate equivalent to the 1% monthly discount rate in the anticipated simple interest accumulation.
Solve the following problems on accumulation and discounting:

7) An investment that lasts 4 months can be made at annual simple interest rate of $i = 20\%$ or at annual compound interest rate of $i' = 20\%$. Which one is more convenient?

8) A capital is invested for 6 months, two options are given: (i) to buy a zero-coupon bond with maturity 6 months earning simple annual yield 14\%; (ii) to buy a zero-coupon bond with maturity 2 months paying simple annual yield 12\%; the reinvestment for the remaining period earns $j$ simple interest per year. Determine $j$ such that the former option is more convenient than the latter.

9) Let the capital $C$ compounded 5\% semi-annually. How long time is needed since the principal be doubled?

10) A bill of 2000€ with maturity 6 months is bank discounted at annual interest rate 4\%. Compute the amount on hand.