1. First Module of Financial Mathematics (TIBILETTI)

Suggested readings and bibliography for the first module


1.1 Basic vocabulary

Financial calculus is based on two types of processes:

- **Accumulation**
- **Discount**

In the accumulation process, money is brought forward in time. For instance, Treasury Bills (T-Bills):

With the discount process, money is brought back in time; for instance, pricing a T-Bill today means transforming an amount of money $S$, which will be available at some future maturity date, into cash today.
We can label a financial operation as

BOTH

an accumulation process and a discount process, according to our perspective.

A T-Bill is interpretable as:

- an accumulation operation for the buyer (the investor);
- a discount operation for the seller (the bank)

It is both practical and convenient to maintain this “dual” perspective to classify financial operations.
**Basic language**

C = Principal = (Invested) Capital

M = Future Value = Final Value or Accumulated Value

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<th>C</th>
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<tr>
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A = Present Value or Discounted Value

S = The amount of money available at a future maturity date in a discount operation. Also called

N = Face Value = Nominal Value or Value at Maturity
**Financial Factors** that link C with M and S with A

There are two equivalent ways to compute $M$

1. Additive way: $M = C + I$, where $I = \text{interests}$
2. Multiplicative way: $M = f \cdot C$

**Note**

1. $I$ are expressed in a currency: euro, $\$, £ … and they are supposed NOT NEGATIVE!
2. $f$ is a pure number

**Accumulation Factor**
There are two equivalent ways to compute $A$

1. Additive way: $A = S - D$, where $D = \text{Discount}$

2. Multiplicative way: $A = \varphi \cdot S$

Note:

$D$ is expressed in a currency: euro, $\$, £ …

$\varphi$ is a pure number

Discount Factor
Capitalization and Discounting usually are used in two different frames:

- **we need to construct a financial contract:**
  for instance, we want to calculate the amount that a firm has to pay at some given maturities in order to pay back a loan;

- **we want to describe, forecast or study the dynamics of an investment:**
  for instance, we want to forecast the performance of a speculative investment undertaken in a period of high market price volatility.

The factor $f$ transfers $C$ forward from date 0 to date $T$;

The factor $\varphi$ transfers backward $S$ from $T$ to 0.

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<td>A</td>
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9
Definitions of Interest rate and Discount rate

**Def.** The amount $i$ earned in 1 year gained for an investment of 1 unit of money is called the **interest rate**

\[
\frac{1}{1+i} = 1 \text{ year}
\]

\[
f(1) = 1 + i
\]
**Def.** The amount $d$ paid as a premium to discount a future amount of 1 unit of money payable in 1 year is called **discount rate**

$$\varphi(1) = 1 \cdot (1 - d)$$
Def. **Conjugate factors:**

\( f \) and \( \varphi \) are called financial conjugate factors

\[ \text{iff} \]

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<th>C</th>
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<tr>
<td>0</td>
<td>1 year</td>
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Capital $C$ can be capitalized at epoch 1 by multiplying $C$ by the accumulation factor:

Future value $S$ can be discounted at epoch 1 by multiplying $S$ by the discount factor:

The equality allows us to find a straightforward link between interest and discount.
Example

\[ d = 20\% \], then:

\[ i = \frac{0.2}{1 - 0.2} = 0.25 > 0.2 \]

\[ i = 25\%. \]
ANNUITY

Annuity is a cash flow

Instalments of the annuity = $R_1, R_2, \ldots, R_n$
The annuity **present value** is the sum of the present values of its instalments:

\[ 0 \]

The annuity **future value** is the **sum of the final values** of the instalments.
Annuity Value at date $t = \text{the sum of the final values (computed at date } t) \text{ of the instalments maturing before } t \text{ with discounted (at time } t) \text{ values of the instalments maturing after } t$: 

REMARKS:

No computation difficulties if we use some standard worksheets.

In the following, we will show how we can drastically simplify these computations in some special cases:

• *amount regularity* (for instance: all are equal)

• *maturities regularity* (for instance: they are periodical, i.e., annual)

• *compound* financial law
1. Simple Interest and Simple Discount

The simple accumulation formula requires the interest to be **proportional** (not only to the principal but also) to the investment lifetime $t$

$$I \text{ is proportional to } C \text{ and } t$$

that is

What’s the meaning of $\alpha$?
**Special case:** $C = 1$ (euro) and $t = 1$ (year)

Thus $\alpha$ is the interest earned on 1 euro over 1 year, that is:

**Def.** $i$ is called (annual) simple interest rate

**Def.** The *Simple Interest factor*
Def. The *Simple Discount* or *Rational Discount* is:

**Note**  $i$  is the simple interest rate
Example 1 Setting interest rate $i$, for instance at $i = 10\% = 0.1$, we determine both:

- the simple interest law with a simple annual interest rate of 10% and
- the simple discount law with an interest rate of 10%.
Example 2 You invest 1000 $ at the simple interest rate of 10% for 4 years. The future value is:

The interest is:

Let us now calculate the present value of 1300 with a maturity date of 2 years via the simple discount process with an interest rate of 15%:

The discount is:
1.2.1 Change of time measure: equivalent rates

If we measure time in **months**, the lifetime $t$ (**in years**) of a financial operation must be multiplied by 12:

Equivalence after $T$ years

\[
\begin{array}{cccc}
1 & 1+\text{monthly rate (12}T) \\
1+iT & \\
0 & T \text{ years}
\end{array}
\]
1.2.2 Floating interest rates

If the interest rate is floating, the future value formula needs to be slightly modified.

Suppose that the *annual rate* is:

\[ i_1 \text{ over a time period of duration } t_1, \]
\[ i_2 \text{ over a time period of duration } t_2 \]
\[ \ldots \ldots \]
\[ i_n \text{ over a time period of duration } t_n, \]

\[
0 \quad t_1 \quad t_1 + t_2 \quad T = t_1 + t_2 + \ldots + t_n
\]
Which is the *equivalent* average flat annual rate $i$?

i.e.

Which is the annual rate $i$ such that the final amount is the same?

\[
\begin{align*}
0 & \quad t_1 & \quad t_1 + t_2 & \quad T = t_1 + t_2 + \ldots + t_n
\end{align*}
\]
\[ i = \frac{i_1 t_1 + \ldots i_n t_n}{T} \]  
Arithmetical Average

Example 3

\[ t_1 = 2 \text{ years and } i_1 = 10\% \]

\[ t_2 = 6 \text{ years and } i_2 = 12\% \]

\[ T = 2 + 6 = 8 \text{ years} \]
1.2.3 An application of simple interests: zero coupon bond and BOT

- The Italian bonds are called Buoni Ordinari del Tesoro (BOT) and are commonly carried out using *simple interest*;
- T-Bills in the US are commonly performed using *bank discount*.

Notation

A zero coupon bond (henceforth zcb):

- $A_0 =$ price of today (at date 0)
- $N =$ nominal value or face value after T years
- $T =$ its maturity date.

This kind of bond does not pay coupons at intermediate maturities

$I = N - P =$ interest paid to the subscribers.

*zb* are the elementary bricks with which modern financial engineering builds many complicated financial tools.
Def. The *simple yield to maturity* $r$ of a zcb is the simple interest rate such that the accumulated amount corresponding to the purchase price is *equal* to the reimbursement value $N$: 

$$0 \quad \text{T years}$$
Hence

The purchase price of the zcb is also the present value - using simple discount - of the reimbursement value, when the rate is given by the simple yield:
Re-sell the bond \( t \) years later (with \( t < T \))

\[ A_t = \text{purchase price at epoch } t \]
The investor plans to sell the bond at epoch $t < T$. $A_t$ depends

\[ r_{t,T} = \text{the current interest rate between } t \text{ and } T \]
\[ A_r = N \cdot \frac{1}{1 + r_{t,T}(T - t)} = \frac{N}{1 + r_{t,T}(T - t)} \]
Suppose we only held the bond for the time period going from 0 to t.

**HOW MUCH IS THE SIMPLE RETURN OF THE INVESTMENT lasting from 0 to t?**
\[ r_{0,t} = \frac{rT - \hat{r}_{i,T}(T-t)}{t \left[ 1 + \hat{r}_{i,T}(T-t) \right]} \]
Remarks

The simple interest $r_{0,t}$ for an investment between 0 and $t$ depends on:

- $r_{0,T} =$ simple interest between 0 and T

- $r_{t,T} =$ simple interest between $t$ and $T$ (Note: in the period from $t$ to $T$ the zcb is no longer in the portfolio!!)

- $T$

If $r_{t,T}$ increases the Numerator decreases and the Denominator increases, so the ratio decreases, so $r_{0,t}$ decreases.
1. Special case: \( r_{t,t} = r_{0,t} = r \)

\[
r_{0,t} = \frac{rT - r(T-t)}{t[1 + r(T-t)]} = \frac{rT - rT + rt}{t[1 + r(T-t)]} =
\]

\[
r_{0,t} = \frac{r}{[1 + r(T-t)]} < r
\]

Conclusion:

The simple interest \( r_{0,t} \) of buying the bond today and selling it at date \( t \) is not equal to \( r \), BUT

\textbf{it is always smaller than } r !
That happens *because* we have used the simple interest accumulation.